DIFFERENTIAL GEOMETRY.

I Preliminaries.
II Space Curves.
III, IV. First & Second Fundamental Forms.
V – VIII Applications Geodesics. V
Surface Mapping VI

$$\dot{x} = \frac{dx}{ds} \quad s: \text{arclength}.$$
$$x' = \frac{dx}{dt}.$$

$\vec{t} = \dot{x}$ unit tangent.

Local & Global Properties.
eg. Gauss–Bonnet theorem.
global problem: macroproperties are related to micro"

Concept of Mapping.

- Coordinates in Euclidean Space. ← just a tool.

$$M \xrightarrow{T} M'$$
$$\{P\} \longrightarrow \{P'\}.$$

Continuous:
topological: T, T^{-1} continuous
points: homomorphic

Geometric properties:
invariant w.r.t. direct congruent transformation / displacement
↳ rigid motion of Cartitian coordinate.

- Geometry & Group Theory.

Def A set ~~of~~ G of mapping is called a group of mapping / a transformation group if

1. Identical Mapping $\in G$
2. if $T \in G$, $T^{-1} \in G$.
3. $\forall T_1, T_2 \in G$, $T_1 \circ T_2 \in G$.

examples: projective geo

Def Equivalence classes: Two Config $\in$ in one and the same equivalance classes / Equivalent w.r.t. a certain group.

Vector in Euclidean Space ?

vectors are just directed segment.

Identity of Lagrange:

* $(a\times b)\cdot(c\times d) = (a\cdot c)(b\cdot d) - (a\cdot d)(b\cdot c)$.

specially: $(a\times b)\times c = (a\cdot c)b - (b\cdot c)a$.

Derive ? ?

Mixed Product, Scalar Triplet Product / Determinant

$$|a\,b\,c| \triangleq a\cdot(b\times c) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Combine them $(a\times b)\times(c\times d) = |a\,b\,d|\,c - |a\,b\,c|\,d$.

Derivative. $(a\cdot b)' = a'\cdot b + a\cdot b'$

$(a\times b)' = a'\times b + a\times b'$

$|abc|' = |a'bc| + |ab'c| + |abc'|$

Theory of Curves

M.

∘ Start: Real ~~Numbe~~ Vector Function — x

(6.1). $x = x(t)$, $x_1 = x_1(t)$, $x_2 = x_2(t)$, $x_3 = x_3(t)$

$t \in [a,b]$

$\boxed{\text{Def}}$ (6.1) is the parametric representation of the point set M.

t: parameter of the representation.

$\boxed{\text{Def}}$

Allowable Parametric Representation $x(t)$.

" ( ) " Transformation $t = t(t^*)$.

→ 不能停

$\boxed{\text{Def}}$ Implicit Function $F(x_1, x_2, x_3) = 0$, $G(x_1, x_2, x_3) = 0$. (6.5)

the relation between the points of M & the different value of t is of minor interest.

(6.5) more general the (6.1)

$\boxed{\text{Def}}$ Simple: ~~a certain point~~ No multiple point of the arc.

⇒ one-to-one corresp

↳ a certain point corrsp to several t.

$\boxed{\text{Def 6.1}}$ An arc of a curve : A point set in space $\mathbb{R}^3$ which can be represented by the allowable representation of an equivalence class.

$\boxed{\text{Def 6.2}}$ Curve : The point set can be represented by an equivalence class of allowable representations of the form (6.1) whose interval I is not assumed to be closed or bounded, but which are such that one always obtain an arc of a curve if the values of the parameter t are restricted to any closed and bound subinterval of I.

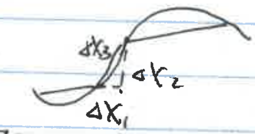
s, s', ds

$\vec{x}, \vec{x}'$ $\dot{x}$

☆

$\boxed{\text{Theorem 9.1}}$ arc C length.

$$s = \int_a^b \sqrt{\sum_{i=1}^{3} \left(\frac{dx_i}{dt}\right)^2}\, dt = \int_a^b \sqrt{x' \cdot x'}\, dt$$



Symbolically, we write $\boxed{ds^2 = \sum_{i=1}^{3} dx_i^{\,2} = dx \cdot dx}$: element of arc / linear element of C

meaning $\{\ s'^2 = \sum_{i=1}^{3} x_i'^{\,2} = x' \cdot x'$

① Note that s is INVARIANT of parametric t / t^*

② s can be used as parameter $x(s)$ – natural parameter.

③ $\dot{\vec{x}} \triangleq \frac{d\vec{x}}{ds}$. $\quad \vec{x}' = \frac{d\vec{x}}{dt}$.

Tangent & Normal Space

$\boxed{\text{Def}}$ $\vec{t}(s) \triangleq \lim_{h\to 0} \frac{\vec{x}(s+h) - \vec{x}(s)}{h} = \frac{d\vec{x}}{ds} = \dot{x}(s)$.

unit tangent vector.

Note :

1. unit : $|\vec{t}|^2 = \vec{t}\cdot\vec{t} = \dot{x}\cdot\dot{x} = \frac{dx}{ds}\cdot\frac{dx}{ds} = 1$
2. other t^* : $\dot{x} = \frac{dx}{dt}\frac{dt}{ds} = \frac{x'}{\sqrt{x'\cdot x'}} = \frac{x'}{|x'|}$
3. tangent : ~~for~~ $\vec{y}(u) = \vec{x} + u\vec{t}$

$\vec{y}(v) = \vec{x} + v\,\vec{x}'$

Osculating Plane.

$P_1, P, P_2 \in$ Curve C.

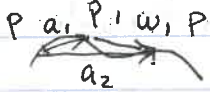
$P_1 \to P$, then $P_1P \to$ tangent.

$P_1, P_2 \to P$ then P_1PP_2 ?? $\to$ Osculating Plane O.

$\det[(z - x), x', x''] = 0. \quad (z \in O)$.

Proof: Let $P_1, P_2 : x(t+h_i)$.

$p \; a_1 \; P_1 \; w, \; P_2$.



$$\vec{a_i} \triangleq \vec{P_0 P_i} = x(t+h_i) - x(t)$$

$$\vec{v_i} \triangleq \frac{\vec{a_i}}{\cancel{\Delta} h_i} = \frac{x(t+h_i) - x(t)}{h_i}$$

$$\vec{w} = \frac{2(\vec{v_2} - \vec{v_1})}{h_2 - h_1}$$

Taylor Expanssion: $x(t+h_i) = x(t) + h_i x' + \frac{h_i^2}{2} x'' + O(h_i^2)$

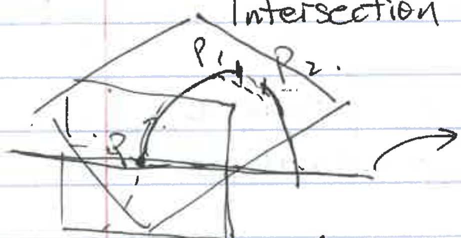
$$\left.\begin{aligned} \vec{v_i} &= x'(t) + \frac{h_i}{2} x'' \\ \vec{w} &= x'' \end{aligned}\right\} \xrightarrow{h_i \to 0} \begin{aligned} \vec{v_i} &= x'(t) \\ \vec{w} &= x''(t) \end{aligned}$$

$\because \vec{v_1}, \vec{w} \in O$

$\therefore$ The plane spanned by $x', x''(t)$ is called Osculating Plane.

$\boxed{\text{Def}}$ Principle ~~A~~ Normal

Intersection of the osculating plane w. corresponding normal plane



principle normal.

12. Curvature $\boxed{\text{Def}}$

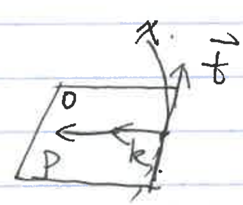
$$\dot{\vec{t}} = \ddot{\vec{x}} \left\langle \begin{array}{l} \text{orthogonal to } \vec{t} \Rightarrow \in \text{Normal Plane} \\ \cancel{\text{orthogona}} \in \text{Osculating Plane} \end{array} \right. \Rightarrow \text{principal normal}.\ \cancel{\text{pricipa}}$$

$$\vec{p}(s) = \frac{\dot{\vec{t}}(s)}{|\dot{\vec{t}}(s)|} \triangleq \underline{\text{unit principal normal}}.$$

$$\kappa(s) = |\dot{\vec{t}}(s)| = \sqrt{\ddot{x}(s) \cdot \ddot{x}(s)} \quad \underline{\text{curvature}}.$$

$$\rho(s) = \frac{1}{\kappa(s)} \quad \text{radius of curvature}.$$

$\boxed{\text{Def}}$ curvature vector $\kappa(s) \triangleq \dot{t}(s)$.

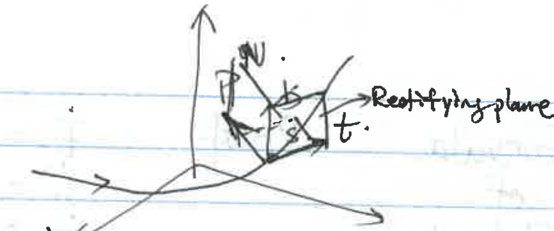


General form of $\kappa = |\dot{t}| = \sqrt{\ddot{x} \cdot \ddot{x}}$

$$\kappa = |\dot{x} \times \ddot{x}| = |x' \times x''| \left(\frac{dt}{ds}\right)^3 = \frac{|x' \times x''|}{|x'|^3}$$

$$= \frac{\sqrt{(x' \cdot x')(x'' \cdot x'') - (x' \cdot x'')^2}}{(x' \cdot x')^{\frac{3}{2}}}.$$

Binormal. Moving trihedron of a curve.



$$b(s) = t(s) \times p(s)$$

where $p(s) = \dfrac{\dot t(s)}{|\dot t(s)|} = \dfrac{\cancel{\kappa}\,\dot t(s)}{\kappa} = \rho\,\dot t(s).$

Note: ~~{t,p} Normal~~

$\vec b$: Osculating Plane

$\vec t$: Normal Plane.

$\vec p$: Rectifying Plane

Torsion. $\boxed{\text{Def}}$ Measure the magnitude and deviation of a curve from the osculating plane.

$$\dot{\vec b}(s) = -\tau(s)\,\vec p(s).$$

$$\tau(s) = -\vec p(s)\cdot\dot{\vec b}(s) \quad \text{— torsion.}$$

Proof. Deviation from Osculating Plane $\triangleq \dot b$

1. 方向: $b\cdot b = 1 \Rightarrow 2b\cdot\dot b = 0 \Rightarrow b \perp \dot b$

$b\cdot t = 0 \Rightarrow \dot b t + b\dot t = 0 \Rightarrow \dot b \perp t.$

$\dot b t = -b\dot t = -b\cdot\kappa p = 0$

$\therefore \dot b \,/\!/\, p \qquad \dot b \triangleq -\tau\vec p \qquad \square$

Note 1. κ - First Curvature / Curvature : $|\dot t| \,/\!/\, \vec p$

τ - Second Curvature / Torsion. $\dot b \,/\!/\, \vec p$?

2. $\tau = \dfrac{|\dot x\ \ddot x\ \dddot x|}{\ddot x\cdot\ddot x}$

Proof $\tau = -\dot b\cdot p = -\dfrac{d(p\times t)}{ds}\cdot p$

$= -\dfrac{d(p\times\dot x)}{ds}\cdot p$

$= -(\dot p\times\dot x + p\times\ddot x)\cdot p$

$= -(\dot p\times\dot x)\cdot p + \cancel{(p\times\ddot x)\cdot p}\ 0$

$p = \rho\ddot x$

$\dot p = \dot\rho\ddot x + \rho\dddot x$

$= \cancel{-(\rho\ddot x\times\dot x)\cdot\rho\ddot x}$

$= -|\rho\ddot x,\ \dot x,\ \dot\rho\ddot x + \rho\dddot x|$

$= \rho^2|\dot x, \ddot x, \dddot x| = \dfrac{|\dot x\ \ddot x\ \dddot x|}{\ddot x\cdot\ddot x}$

(Recap: $\rho = |\ddot x|$

Formula of Frenet.

· Proof

$\dot{\vec{t}} = \kappa \vec{p}$

$\dot{\vec{b}} = -\tau \vec{p}$

$\dot{\vec{p}} \overset{?}{=} \cancel{2\kappa\vec{p}} - \kappa\vec{t} + \tau\vec{b}$

$$\boxed{\begin{pmatrix} \dot{t} \\ \dot{p} \\ \dot{b} \end{pmatrix} = \begin{pmatrix} & \kappa & \\ -\kappa & & \tau \\ & -\tau & \end{pmatrix} \begin{pmatrix} t \\ p \\ b \end{pmatrix}}$$

① Direction:

From $\vec{p} \cdot \vec{p} = 1 \Rightarrow \vec{p} \cdot \dot{\vec{p}} = 0 \Rightarrow \vec{p} \perp \dot{p} \Rightarrow \dot{\vec{p}} = a\vec{t} + c\vec{b}$

② Magnitude. $* \vec{t} \Rightarrow a = t \cdot \dot{p} \quad = -\dot{t}p = -\kappa$

$* \vec{b} \Rightarrow c = b \cdot \dot{p} = -\dot{b}p = -(-\tau)$.

Def General / Cylindrical Helix: tangent make a constant angle with a fixed line in space.

Theorem A twisted ~~class~~ curve of class $r \geq 3$ with non-vanishing curvature is a general helix iff at all points, the ratio $\frac{\tau(s)}{\kappa(s)} = \text{Const}$

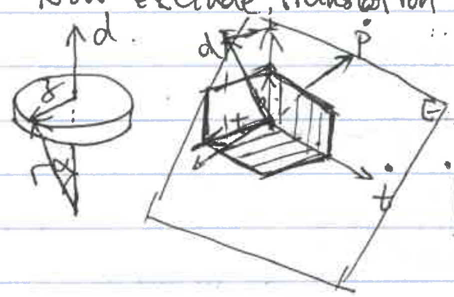
Motion of the trihedron, vector of Darboux.

A point moves along a curve $C \Rightarrow$ corresponding trihedron makes a motion as well. $\Rightarrow$ imbed the $t.p.b$ in a rigid body K that performs same motion as the trihedron $\Rightarrow$ kinematic interpretation

Theorem 16.1 Any motion of a rigid body in space is, at every instant, an (infinitesimal) screw motion.

Note: any point $P \in$ moving body K, is a circular helix.

Now exclude translation and only focus on rotation – represented by rotation vector.



Question: Given a curve and $P \in C$, what is the rotation vector $\vec{d}$ at any moment?

Theorem 16.2 $\vec{d} = \tau\vec{t} + \kappa\vec{b}$

Note: 1. Fernet Formula now becomes.

$\dot{t} = d \times t$; $\dot{p} = d \times p$; $\dot{b} = d \times b$

2. See Fig 20

3. Prove urself.

想象？$\vec{t}, \vec{p}, \vec{b}$
在Fig 19 的哪里？
why？$\vec{t} \perp \vec{d}$？
证明？

Spherical Image of a Curve

Investigate the vectors of the moving trihedron of a curve $C: x(s)$.
Assume the vectors undergo a parrellel displacement thus origin bound at
There fore, p, t, b are on unit sphere. and the trajectories are curves on sphere S. S_T, S_P, S_B: ~~tangent / principle normal / binormal indicatrix~~.

from the ds definition $(ds)^2 = dx^\rho \cdot dx^\rho$

$$(ds_T)^2 = \dot{t}\cdot\dot{t}\,(ds)^2 = \kappa^2\, p\cdot p\,(ds)^2 = \kappa^2 ds^2$$

$$ds_p^2 = \dot{p}\cdot\dot{p}\,(ds)^2 = (-\kappa t+\tau b)(-\kappa t+\tau b)\,ds^2 = (\kappa^2+\tau^2)\,ds^2.$$

$$ds_B^2 = \dot{b}\cdot\dot{b}\,(ds)^2 = (-\tau p)^2 ds^2 = \tau^2 ds^2.$$

$$\Rightarrow \quad \kappa = \frac{ds_T}{ds} \qquad \tau = \frac{ds_B}{ds}.$$

推论: $ds_p^2 = ds_T^2 + ds_B^2$ Equation of Lancret.

Note: 1. Diff curves might have same spherical image.
eg. 同心圆

Canonical Representation

Shape of a Curve in the neighborhood of any of its point.

$\boxed{\text{Def}}$ Canonical representation of the curve C.

$$x_0(s) \approx \left(s,\ \frac{\kappa_0}{2}s^2,\ \frac{\kappa_0\tau_0}{6}s^3\right).$$

Taylor Expassion. $x(s) = x(0) + s\cdot\dot{x}(0) + \frac{s^2}{2}\ddot{x}(0) + \frac{s^3}{6}\dddot{x}(0) + o(s^3)$

$$\dot{x} = t$$

$$\ddot{x} = \dot{t} = \kappa p$$

$$\dddot{x} = \dot{\kappa}p + \kappa\dot{p} = \dot{\kappa}p + \kappa(-\kappa t + \tau b) = \dot{\kappa}p - \kappa^2 t + \kappa\tau b.$$

Set up Cartesian coordinate as $t(0) = (1.0.0)$ $p(0) = (0.1.0)$ $b(0) = (0.0.1)$
$\qquad x_1 \qquad x_2 \qquad x_3.$
origin at $x(0)$.

By substitution,

$$x(s) = \begin{pmatrix}0\\0\\0\end{pmatrix} + \begin{pmatrix}1\\0\\0\end{pmatrix}s + \frac{\kappa}{2}\begin{pmatrix}0\\1\\0\end{pmatrix}s^2 + \frac{s^3}{6}\left[\dot{\kappa}\begin{pmatrix}0\\1\\0\end{pmatrix} - \kappa^2\begin{pmatrix}1\\0\\0\end{pmatrix} + \kappa\tau\begin{pmatrix}0\\0\\1\end{pmatrix}\right]$$

$$\Rightarrow \quad x_1(s) = s - \cancel{\frac{\kappa^2}{6}s^3} \ \cancel{+\,o(s^3)}$$

$$x_2(s) = \frac{\kappa}{2}s^2 + \cancel{\frac{s^3}{6}\dot{\kappa}} \ \cancel{+\,o(s^3)}$$

$$x_3(s) = \frac{s^3}{6}\kappa\tau \ \cancel{+\,o(s^3)}.$$

ignore non-leading term:

$$x(s) \approx \left(s,\ \frac{\kappa}{2}s^2,\ \frac{\kappa\tau}{6}s^3\right)^T.$$

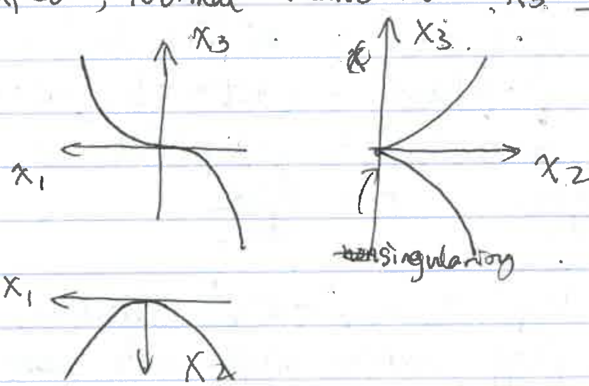
Note: 1. we can define the point C corresponds to $S=0$ ARBITRARILY. thus, Taylor expanssion at $s=0$ does not lose generality.

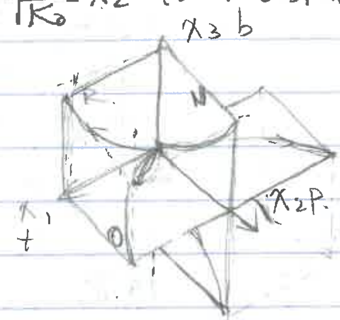
2. Consider the neighbourhood curve projecting to 3 planes:

x_1x_2, $x_3=0$, Osculating Plane O: $x_2 = \frac{k_0}{2} x_1^2$ (quadratic parabola)

x_1x_3 $x_2=0$, Rectifying Plane R: $x_3 = \frac{k_0 \tau_0}{6} x_1^3$ (cubical parabola)

x_2x_3 $x_1=0$, Normal Plane N: $x_3 = \frac{\sqrt{2}}{3} \frac{\tau_0}{\sqrt{k_0}} x_2^{3/2}$ (semi-cubical parabola).





see Fig 23.

Contact, Osculating Sphere.

$\boxed{\text{Def}}$ Contact: 1° Curve–curve $\alpha_i(s) = \beta_i(s^*)$ $\frac{d^\mu \alpha_i}{ds^\mu} = \frac{d^\mu \beta_i}{ds^{*\mu}}$ $\left(\begin{array}{l} i=1,2,3 \\ \mu = 1,2 \cdots m \end{array}\right)$

$\frac{d^{m+1}\alpha}{d\alpha^{m+1}} \neq \frac{d^{m+1}\beta}{ds^{*m+1}}$. — m-order.

2° Curve to Plane. (1) $\exists$ $C^* \in S$, C, C^* has contact order m.,
(2) $\nexists$ $C^* \subset S$, $C.C^*$ has order $(m+1)$

19.4b $\boxed{\text{Theorem } 19.1}$ $\forall P \in C$, C at P has contact of $\geq$2nd order with its osculating plane.

Lemma $C: x(s)$. have a point $P_0: S=S_0$ in common with surface S which has representation $G(x_1.x_2.x_3)=0$: Then C has contact of order m with S at P_0 iff $p(S) = G(\alpha_1(S), \alpha_2(S), \alpha_3(S))$ and its derivative up to m-th vanish at P_0 while m+1-th does not vanish.

$$p(S_0)=0 \quad \left.\frac{d^\mu p}{ds^\mu}\right|_{s=s_0} = 0. \quad (\mu = 1 \cdots m) \quad \frac{d^{m+1} p}{ds^{m+1}} \neq 0.$$

$\boxed{\text{Theorem } 19.3}$ $C \cap S \ni P_0$ ~ m order, $\begin{cases} \text{m even: C pierces S.} \\ \text{m odd: lies on one side} \end{cases}$

Proof: consider p(S) ~~and~~ with Taylor Expanssion till m+1

$\swarrow (x-a)\cdot(x-a) - \rho^2 = 0$

$\boxed{\text{Theorem}}$ 1° The center of any sphere which has contact of $\geq$ 1st order with a curve C at P lies in the normal plane to C at P:

$$\vec{a} = \vec{x} + \alpha\vec{p} + \beta\vec{b} \quad \leftarrow \text{by } \frac{d\rho}{ds} = 0.$$

2° ~~$\vec{a}$ has~~ The center of any sphere ~~which~~ which has contact of 2nd order ·· POLAR AXIS

$$\vec{a} = \vec{x} + \rho\vec{p} + \beta\vec{b}$$

passing through center of curvature.

3° ---- 3rd order: osculating sphere.

$$\vec{a} = \vec{x} + \rho\vec{p} + \dot{\rho}/\tau\, \vec{b}$$

Natural Equation of a Curve

Goal: Develop equation/form that is independent of the coordinate, parameter; ~~with~~ to with direct congruent transformation (except for position. (s, τ, κ).

$\boxed{\text{Theorem 20.1}}$ $\tau(s)$, $\kappa(s) > 0$ be continuous functions, $I: 0 \leq s \leq a$. Then there exists one and only one ~~arc~~ $x(s)$ of a curve, determined up to a direct congruent transformation whose curvature/torsion are given by κ, τ.

Note: 1. it does ~~not say~~ suggest inverse. $\forall\, \tau, \kappa \Rightarrow x$.

?? 20.8 → limit?

2. the proof is about function infinity. ~~don't and~~

~~3. The representation x can be obtained $x(s) = x_0 + \int_0^s t(\sigma)\,d\sigma$.~~

3. it's defined by 3 ordinary linear equation.

$$(\dot{\vec{v}}_i) = C(\vec{v}_i). \quad i = 1, 2, 3.$$

where $\vec{v}_1 = \vec{t}$, $\vec{v}_2 = p$, $\vec{v}_3 = b$. $C = \begin{pmatrix} & \kappa & \\ -\kappa & & \tau \\ & -\tau & \end{pmatrix}$

~~Involutes & Evolutes~~

eg 1. what is $\rho = r = \text{const}$, $\tau = 0$.

Sol: Circle.

Lemma: ~~dx_1~~ Plane curve fact. $x(S)$ be $\overset{t}{\angle\alpha} \rightarrow x$, then:

$\vec{t} = (\cos\alpha, \sin\alpha)$ $d\vec{t}/d\alpha = (-\sin\alpha, \cos\alpha) \Rightarrow \vec{t}\cdot\frac{d\vec{t}}{d\alpha} = 0 \Rightarrow d\vec{t}/d\alpha \perp \vec{t}$

$\Rightarrow d\vec{t}/d\alpha = \pm p$ (plane curve). $\dot{t} = \frac{dt}{d\alpha}\cdot\frac{d\alpha}{ds} = \kappa p$

$\therefore \left|\frac{d\alpha}{ds}\right| = \kappa$, choosing orientation, $\kappa = \frac{d\alpha}{ds}$. $\triangle$.

$dx_1 \overset{\text{by def}}{=} \cancel{d(s\cos\alpha)}\ (ds)\cos\alpha = r\cos\alpha\, d\alpha.$

$dx_2 = r\sin\alpha\, d\alpha$

$\Rightarrow x_1 = \int dx_1 = \int_0^{\alpha_0} r\cos\alpha\, d\alpha = r\sin\alpha.$

$x_2 = r(1-\cos\alpha).$ $\Big\} \Rightarrow x_1^2 + (x_2 - r)^2 = r^2.$

eg2. What does $\begin{cases} s\rho = c^2 \\ \tau = 0 \end{cases}$ yield? (Spiral of Cornu)

Sol: this time we have relation ~~that~~ s. so we might wanna convert α to s

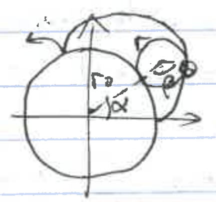
$dx_1 = \cos\alpha \, ds$. $\quad x_1 = \int_0^s \cos\alpha \, d\sigma$

We need $\alpha(s)$ → From $\kappa = \frac{d\alpha}{ds}$,

$$\alpha = \int_0^s \kappa \, d\sigma = \int_0^s \frac{\sigma}{c^2} d\sigma = \frac{s^2}{2c^2}.$$

$$\therefore x_1 = \int_0^s \cos\frac{\sigma^2}{2c^2} d\sigma \overset{\Phi = \frac{\sigma^2}{2c^2}}{=} \frac{c}{\sqrt{2}} \int_0^{\Phi} \frac{\cos\alpha}{\sqrt{\alpha}} d\alpha.$$

cannot express analytically. But lim $\int_0^\infty \frac{\cos\alpha}{\sqrt{\alpha}} d\alpha = \int_0^\infty \frac{\sin\alpha}{\sqrt{\alpha}} d\alpha = \sqrt{\frac{\pi}{2}}$.

eg3: 

Natural Equation of ordinary epicycloid?

$$\beta = \frac{r_0 \alpha}{r} \Rightarrow$$

$$x_1 = (r_0+r)\cos\alpha + r\cos(\pi - \alpha - \beta)$$

$$= (r_0+r)\cos\alpha - r\cos\left(\frac{r_0+r}{r}\alpha\right)$$

$$x_2 = (r_0+r)\sin\alpha - r\sin\left((r_0+r)/r \, \alpha\right)$$

then Calculate $s = \int_0^s \sqrt{dx_1^2 + dx_2^2} = a\cos\frac{r_0}{2r}\alpha$.

$$a = 4\frac{r}{r_0}(r_0+r)$$

in this way, we cancel x_1, x_2, which ~~is~~ depends on coordinate.

From calculation $\rho = b\sin\frac{r_0}{2r}\alpha$, $b = \frac{4r}{r_0+2r}(r_0+r)$

$\Rightarrow$ 消 $\alpha \Rightarrow \frac{s^2}{a^2} + \frac{\rho^2}{b^2} = 1$, $\tau = 0$.

Involutes and Evolutes

$\boxed{\text{Def}}$ Tangent surface: surface $\vec{x}$ and $\vec{t}$ spans.

$$\vec{y}(s,u) = \vec{x}(s) + u\vec{t}(s)$$

Note: u is distance of point p to the tangent

$\boxed{\text{Def}}$ Involutes of curve are curves on the corresponding tangent surface which are orthogonal to the generating tangent.

$$\vec{z}(s) = \vec{x}(s) + (c-s)\vec{t}(s)$$

Proof. $z(s) = x(s) + u(s)t(s)$

From def of involutes: $\dot{z} \perp \dot{x}$

$$(\dot{x} + u\dot{t} + \dot{u}t)(\dot{x}) = 0$$

$$\Rightarrow (t + u \cdot \kappa p + \dot{u}t)t = 0$$

$$\Rightarrow (1+\dot{u})\, t \cdot t = 0 \Rightarrow \dot{u} = -1. \quad u = c - s.$$

$\boxed{\text{Def}}$ Evolute of C : Let a curve C be given and determine a curve C^* s.t. the given C is an involute of C^*.

Find form : $C: \vec{x}(s)$.

$C^*: \vec{y}(s) = \vec{x}(s) + q(s)\,\vec{a}(s)$,

where $\vec{a}$ is ~~tan~~ unit vector, tangent of $\vec{y}(s)$. (see Fig 29).

$$\dot{y} = \beta \vec{a} \qquad \dot{y} \,/\!/\, \vec{a}$$

$$\overset{\circ}{x} + \overset{\circ}{q} a + q \overset{\circ}{a} = t + \overset{\circ}{q} a + q \overset{\circ}{a}$$

$\vec{a}, t \perp a \Rightarrow \quad \beta = \overset{\circ}{q}$

$\therefore \quad t + q\overset{\circ}{a} = 0.$

$\therefore \quad a \in$ Normal plane.

$\therefore \quad a = p \sin\alpha + b\cos\alpha$. $\frac{d}{ds}$.

$$t + q[(-\kappa t + \tau b)\sin\alpha + \dot{\alpha} p \cos\alpha - \tau p\cos\alpha - \dot{\alpha} b \sin\alpha] = 0, \ \forall s.$$

$\therefore$ coeff of t, b, p must vanish.

$$\therefore \begin{cases} 1 - \kappa q \sin\alpha = 0 & \Rightarrow \quad q = \rho/\sin\alpha, \\ (\dot{\alpha} - \tau)\cos\alpha = 0 & \forall\alpha \\ (\tau - \dot{\alpha})\sin\alpha = 0 \end{cases} \Rightarrow \dot{\alpha} = \tau \text{ must hold.}$$

$$\alpha = \int_0^s \tau(s)\,ds + C_0$$

綜上, $\vec{y}(s) = x(s) + \rho(s)[\vec{p}(s) + \vec{b}(s)\cot\alpha(s)]$, where $\alpha =$

one C_0 corresponds to one curve of evolute.

Bertrand Curves.

$\boxed{\text{Def}}$ Two curves which, at any of their points, have a common $\vec{p}$ are called Bertrand Curves.

III CONCEPT OF A SURFACE. 1ST FUNDAMENTAL FORM. FOUNTATIONS OF TENSOR-CALCULUS.

Concept of a surface in diff geometry.

? Determinant of order R "of matrix J"

$\boxed{\text{Def}}$ ~~$\vec{x}(u^1, u^2) = x_1(u_1;$~~

para-repr. $\vec{x}(u^1,u^2) = (x_1(u^1,u^2), x_2(u^1,u^2), x_3(u^1,u^2))$.

Differential assumption: $J = \begin{pmatrix} \frac{\partial x_1}{\partial u^1} & \frac{\partial x_1}{\partial u^2} \\ \frac{\partial x_2}{\partial u^1} & \frac{\partial x_2}{\partial u^2} \\ \frac{\partial x_3}{\partial u^1} & \frac{\partial x_3}{\partial u^2} \end{pmatrix}$ is of rank 2.

denote $\vec{x}_\alpha = \frac{\partial \vec{x}}{\partial u^\alpha}$ $x_{\alpha\beta} = \frac{\partial^2 \vec{x}}{\partial u^\alpha \partial u^\beta}$.

~~Allowable Representation Assumption.~~

(1). $\vec{x}(u^1,u^2)$ is of class $r \geq 1$ in B. Each point of the set M, corresponds to just one ordered pair $(u^1 u^2)$ in B.

(2). J is of rank 2 everywhere in B ($\forall u^1 u^2$).

Allowable Coordinate Transformation $u^\alpha = u^\alpha(\bar{u}^1, \bar{u}^2)$.

(1). The functions $u^\alpha(\bar{u}^1, \bar{u}^2)$ are defined in $\bar{B}$ s.t. the corresponding range of values includes B.

(2). u^α are of class $r \geq 1$ and is a one-to-one transformation.

(3). $D = \frac{\partial u^1 u^2}{\partial \bar{u}^1 \bar{u}^2} = \begin{vmatrix} \frac{\partial u^1}{\partial \bar{u}^1} & \frac{\partial u^2}{\partial \bar{u}^1} \\ \frac{\partial u^1}{\partial \bar{u}^2} & \frac{\partial u^2}{\partial \bar{u}^2} \end{vmatrix} \neq 0$. $\forall (\bar{u}^1, \bar{u}^2)$

Remark 1. (2) (3) are independent.

eg. $u = (e^{\bar{u}^1} \cos \bar{u}^2, e^{\bar{u}^1} \sin \bar{u}^2)$.

$u = ((\bar{u}^1)^3, \bar{u}^2)$. ☆

2. Implicit Function: $G(x_1, x_2, x_3) = 0$.

$\boxed{\text{Def}}$ Portion of a surface. A point set in $\mathbb{R}^3$ which can be represented by the allowable repr[e]sentations of an equivalence class.

Further Remark. The fact that at certain points of B the J rank < 2 may be either due to special choice of the representation, or the geometric shape itself.

Curves on a Surface, Tangent Plane to a Surface

Curve on a Surface.

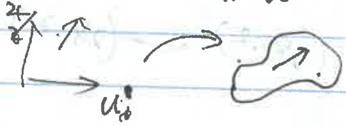
$S: x(u^1, u^2)$ ① $u^1 = u^1(t)$ $u^2 = u^2(t)$

② $u^2 = u^2(u^1)$

③ $h(u^1, u^2) = 0$

The direction of the tanget to a curve $u^1(t)\, u^2(t)$ on $x(u^1, u^2)$ is determined by $x' = \frac{\partial x}{\partial t} = \frac{\partial x}{\partial u^1}\frac{\partial u^1}{\partial t} + \frac{\partial x}{\partial u^2}\frac{\partial u^2}{\partial t} = \vec{x}_1 \cdot u^{1\prime} + \vec{x}_2 \cdot u^{2\prime}$

$\Rightarrow$ 1. $\vec{x}_1, \vec{x}_2$ spans the tangent ~~surf~~ Plane.



2. $u^{1\prime}\ u^{2\prime}$ depends on ~~coordi~~ choice of t.

3. Tangent Plane $E(P)$: $y(q_1 q_2) = \vec{x} + q_1 \vec{x}_1 + q_2 \vec{x}_2$.

$\Leftrightarrow |(y - x), x_1, x_2| = 0$.

if implicit function: $\sum_{i=1}^{3} (y_i - x_i) \frac{\partial G}{\partial x_i} = 0$.

Note $\left(\frac{\partial G}{\partial x_1}\ \frac{\partial G}{\partial x_2}\ \frac{\partial G}{\partial x_3}\right) \perp E(P)$.

First Fundamental Form. Concept of Riemannian Geometry

$\boxed{\text{Def}}$ Any curves on a surface $S: \vec{x}(u^1, u^2)$ can be represented in the form:

$u^1 = u^1(t)$, $u^2 = u^2(t)$.

The element of arc of such a curve

(27.2) $ds^2 = dx \cdot dx = (\vec{x}_1 du^1 + \vec{x}_2 du^2)^2 = \begin{pmatrix} du^1 \\ du^2 \end{pmatrix}^T \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} du^1 \\ du^2 \end{pmatrix}$

$= g_{\alpha\beta}\, du^\alpha du^\beta$.

where $x_\alpha \cdot x_\beta = g_{\alpha\beta}$. $= g_{11}(du^1)^2 + 2g_{12}\, du^1 du^2 + g_{22}(du^2)^2$.

~~27~~ The quadratic form (27.2) is called 1st fundamental form.

Remarks: 1. First Fundamental Form enables us to measure arc length, angles, area of surfaces. It defines a 'metric' on a surface.

2. $g_{\alpha\beta}$ are components of a tensor — metric/fundamental tensor

3. In general cases, $ds^2 = g_{\alpha\beta}\, du^\alpha du^\beta$ $(\alpha, \beta = 1 \cdots n)$

4. Each surface defines a 1st FF Function.

Properties of the 1st FF

$\boxed{\text{Theorem}^{28.1}}$ At regular points of a surface, the 1st-FF > 0.

Regular Point: $J = \begin{pmatrix} \frac{\partial x_1}{\partial u^1} & \frac{\partial x_1}{\partial u^2} \\ \frac{\partial x_2}{\partial u^1} & \frac{\partial x_2}{\partial u^2} \\ \frac{\partial x_3}{\partial u^1} & \frac{\partial x_3}{\partial u^3} \end{pmatrix}$ $\exists$ 2nd-determinant > 0.

$\Rightarrow \begin{vmatrix} \frac{\partial x_1}{\partial u^1} & \frac{\partial x_1}{\partial u^2} \\ \frac{\partial x_2}{\partial u^1} & \frac{\partial x_2}{\partial u^2} \end{vmatrix}^2 + \left|\quad\right|^2 + \left|\quad\right|^2 > 0$

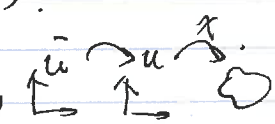
? ? The previous sum is $= \det(g_{\alpha\beta}) = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} > 0$.

$\therefore$ Positive Definite.

$\boxed{\text{Theorem}}$ 28.2. 1st-FF under a transformation of coordinate.

Consider a new coordinate $\bar{u}^1\ \bar{u}^2$: $u^\alpha = u^\alpha(\bar{u}^1\ \bar{u}^2)$.

Then, the coeff $\bar{g}_{\mu\nu}$ of this form w.r.t $\bar{u}^1.\bar{u}^2$ are related as



(28.3) ← (28.3)
$$\bar{g}_{\mu\nu} = g_{\alpha\beta}\frac{\partial u^\alpha}{\partial \bar{u}^\mu}\frac{\partial u^\beta}{\partial \bar{u}^\nu} \qquad g_{\alpha\beta} = \bar{g}_{\mu\nu}\frac{\partial \bar{u}^\mu}{\partial u^\alpha}\frac{\partial \bar{u}^\nu}{\partial u^\beta}$$

$$= \sum_\alpha\sum_\beta \frac{\partial u^\alpha}{\partial \bar{u}^\mu}\cdot\frac{\partial u^\beta}{\partial \bar{u}^\nu}\cdot g_{\alpha\beta}.$$

Proof: $u^\alpha = \frac{\partial u^\alpha}{\partial \bar{u}^\mu} du^\mu$.

$$ds^2 = g_{\alpha\beta}du^\alpha du^\beta = \bar{g}_{\mu\nu}d\bar{u}^\mu d\bar{u}^\nu.$$

$$\Rightarrow g_{\alpha\beta}\frac{\partial u^\alpha}{\partial \bar{u}^\mu}\cancel{d\bar{u}^\mu}\frac{\partial u^\beta}{\partial \bar{u}^\nu}\cancel{d\bar{u}^\nu} = \bar{g}_{\mu\nu}\cancel{d\bar{u}^\mu}\cancel{d\bar{u}^\nu}$$ ← 能消吗? 不能!!

$$\Rightarrow \cancel{\bar{g}_{\mu\nu} = g_{\alpha\beta}\frac{\partial u^\alpha}{}}$$

$$= g_{\alpha\beta}\frac{\partial u^\alpha}{\partial \bar{u}^\mu}\frac{\partial u^\beta}{\partial \bar{u}^\nu}d\bar{u}^\mu d\bar{u}^\nu = \bar{g}_{\mu\nu}d\bar{u}^\mu d\bar{u}^\nu.$$

根据形式比对: $\bar{g}_{\mu\nu} = g_{\alpha\beta}\frac{\partial u^\alpha}{\partial \bar{u}^\mu}\frac{\partial u^\beta}{\partial \bar{u}^\nu}$.

$\boxed{\text{Theorem}}$ 28.3. If $u^1.u^2$ undergo an allowable trans. $u^\alpha = u^\alpha(\bar{u}^1, \bar{u}^2)$; then:

(28.4)
$$\bar{g} = D^2 g, \quad g = \bar{D}^2\bar{g}.$$

where $\bar{g}$ is the discriminant(判别式) of the 1st-FF w.r.t. $\bar{u}^1.\bar{u}^2$ ($g_{11}g_{22}-g_{12}^2$)

$$D = \det\frac{\partial(u^1.u^2)}{\partial(\bar{u}^1\ \bar{u}^2)} = \begin{vmatrix} \frac{\partial u^1}{\partial \bar{u}^1} & \frac{\partial u^1}{\partial \bar{u}^2} \\ \frac{\partial u^2}{\partial \bar{u}^1} & \frac{\partial u^2}{\partial \bar{u}^2} \end{vmatrix} \qquad \bar{D} = \frac{\partial(\bar{u}^1.\bar{u}^2)}{\partial(u^1\ u^2)}$$

Proof: $\bar{g} = \bar{g}_{11}\bar{g}_{22} - \bar{g}_{12}^2 = g_{\kappa\lambda}g_{\mu\nu}\frac{\partial u^\kappa}{\partial \bar{u}^1}\frac{\partial u^\nu}{\partial \bar{u}^2}\left(\frac{\partial u^\lambda}{\partial \bar{u}^1}\frac{\partial u^\mu}{\partial \bar{u}^2} - \frac{\partial u^\lambda}{\partial \bar{u}^2}\frac{\partial u^\mu}{\partial \bar{u}^1}\right)$.

展开求和项,消0. 实现手法:观察发现 $\lambda=\mu$ 时 $=0$.

故可展开为 $\bar{g} = (g_{\kappa 1}g_{2\nu} - g_{\kappa 2}g_{1\nu})\frac{\partial u^\kappa}{\partial \bar{u}^1}\frac{\partial u^\nu}{\partial \bar{u}^2}\left(\frac{\partial u^1}{\partial \bar{u}^1}\frac{\partial u^2}{\partial \bar{u}^2} - \frac{\partial u^1}{\partial \bar{u}^2}\frac{\partial u^2}{\partial \bar{u}^1}\right)$

又观察, $\kappa=\nu$ 时 $=0$,

$$\Rightarrow \bar{g} = (g_{11}g_{22}-g_{12}^2)D^2 = D^2 g$$

Contravariant and Covariant Vectors. (Intro to tensor Calculus).

~~Cond~~ Consider General cases. n-dim: $u^\alpha = u^\alpha(\bar{u}^1 \cdots \bar{u}^n)$.

$\bar{u} \underset{h}{\overset{f}{\rightleftarrows}} u$ Chain rule: $\begin{cases} \dfrac{\partial u^\alpha}{\partial \bar{u}^\gamma} \dfrac{\partial \bar{u}^\gamma}{\partial u^\beta} = \delta_\beta{}^\alpha \\ \dfrac{\partial \bar{u}^\alpha}{\partial u^\gamma} \dfrac{\partial u^\gamma}{\partial \bar{u}^\beta} = \delta_\beta{}^\alpha \end{cases}$

$\Leftrightarrow$ $d\bar{u}^\beta = \dfrac{\partial \bar{u}^\beta}{\partial u^\alpha} du^\alpha$; $du^\gamma = \dfrac{\partial u^\gamma}{\partial \bar{u}^\beta} d\bar{u}^\beta$.

$\Rightarrow$ This is a linear system of differentials. Thus, $u, \bar{u}$ induces a homogeneous linear transformation, coeff are functions of the coordinate,
rule
not constant

Def Contravariant. Let n-tuple $a^1 \cdots a^n$ $\in \mathbb{R}$ be associated with a point P of an n-dim Riemannian space with a coordinate $u^1 \cdots u^n$. Furthermore, Let there be associated with P an n-tuple of real number $\bar{a}^1 \cdots \bar{a}^n$ w.r.t. coordinate system $\bar{u}^1 \cdots \bar{u}^n$ which can be obtained from the coordinates u^α by an allowable transformation. If these numbers satisfy the relations

(29.4) $\Rightarrow$ $\bar{a}^\beta = \dfrac{\partial \bar{u}^\beta}{\partial u^\alpha} a^\alpha$. $\Leftrightarrow$ $a^\gamma = \dfrac{\partial \bar{u}^\gamma}{\partial u^\beta} \bar{a}^\beta$.

then we say a contravariant tensor of first order or contravariant vector at P is given. $a^1 \cdots a^n$, $\bar{a}^1 \cdots \bar{a}^n$ are called the component of this vector in their respective coordinate systems. (du^α). This vector will be denoted as a^α, $\bar{a}^\alpha$. (29.4) is called transformation behavior, indicated by a superscript.

Remark 1. $\forall$ n real number can be taken as components of a contravariant vector w.r.t u^α at P, then $\bar{a}^\alpha$ are determined by (29.4)

2. Euclidean space is a special Riemannian space where free vector is appropriate (linear trans ~~rigid~~), but in general cases of Riem— bound vector is appropriate.

Def Geometric object (~~not related to cor~~ $\Rightarrow$ contravariant) if the following holds:

1). w.r.t every allowable coordinate system, one & only one ordered N real number is given (component of the geo obj in respective coord).

2). A law is given which permits the representation of the component of obj in terms of

a) the component of this obj wrt any u^α.

b) the value at P of the function involved in the u^α trans and derivatives

只是某些 function family

Remarks: 1. Specially, ~~sa~~scalar / ~~ge~~ invariants $\in$ GeoObj. $N=1$
2. Contravariant vectors $\in$ GeoObj, which transform by (29.4) $N=n$.
3. Covariant:

$\boxed{\text{Def}}$ Covariant $\left\{ \begin{array}{l} b_1 \cdots b_n \sim P \text{ in } u^1 \cdots u^n, \quad u^\alpha = u^\alpha(\bar{u}^1 \cdots \bar{u}^n). \\ \bar{b}_1 \cdots \bar{b}_n \quad \bar{u}^1 \cdots \bar{u}^n \end{array} \right.$

cova / contra

(29.5) ~~to~~ Relations: $\bar{b}_\beta = b_\alpha \dfrac{\partial u^\alpha}{\partial \bar{u}^\beta} \Leftrightarrow b_\gamma = \bar{b}_\beta \dfrac{\partial \bar{u}^\beta}{\partial u^\gamma}$.

(29.4) in contrast $\bar{a}^\beta = a^\alpha \dfrac{\partial \bar{u}^\beta}{\partial u^\alpha} \quad a^\gamma = \bar{a}^\beta \dfrac{\partial u^\gamma}{\partial \bar{u}^\beta}$.

Remark: 1. How quantities b_α must behave under a coordinate trans, s.t. $b_\alpha a^\alpha$ is invariant? (a^α is contravariant)
2. Special case: $ds^2 = g_{\alpha\beta}\, du^\alpha du^\beta$, $\leftarrow$ second order.
ds^2: invariant
$du^\alpha du^\beta$: contravariant; $g_{\alpha\beta}$: covariant.

? any vector field $\Leftrightarrow$ cov? $\Updownarrow$ $\exists$ scalar functn?

eg1 做功 $W = Fx$
$dW = p_\alpha du^\alpha$ $\left\{ \begin{array}{l} dW: \text{invariant} \\ du^\alpha: \text{contravariant} \\ p_\alpha: (\text{Force}) \text{ covariant} \end{array} \right.$

eg2. $\phi(u^1 \cdots u^n) \in \mathbb{R}$, and is invariant; $\Leftrightarrow$ scalar function. the $\dfrac{\partial \phi}{\partial u^\alpha} \in$ covariant.
$$\frac{\partial \phi}{\partial \bar{u}^\beta} = \frac{\partial \phi}{\partial u^\alpha} \frac{\partial u^\alpha}{\partial \bar{u}^\beta}.$$

Contravariant Covariant & Mixed Tensor

Introduce contravariant / cov tensor of arbitrary order.

$\boxed{\text{Def}}$ 2nd-order contravariant tensor. n^2: $a^{\alpha\beta}$ $(\alpha, \beta = 1 \cdots n)$ in $u^1 \cdots u^n$ at P.
$\bar{a}^{\gamma\kappa}$ $\quad \bar{u}^1 \cdots \bar{u}^n$

Relation: $\bar{a}^{\gamma\kappa} = a^{\alpha\beta} \dfrac{\partial \bar{u}^\gamma}{\partial u^\alpha} \dfrac{\partial \bar{u}^\kappa}{\partial u^\beta}$; $a^{\sigma\tau} = \bar{a}^{\gamma\kappa} \dfrac{\partial u^\sigma}{\partial \bar{u}^\gamma} \dfrac{\partial u^\tau}{\partial \bar{u}^\kappa}$

Remark: 1. It's done by considering $J = a^{\alpha\beta} b_\alpha c_\beta$,
J: invariant; b_α: covariant.

$\boxed{\text{Def}}$ $a_{\alpha\beta}$; $\bar{a}_{\gamma\kappa}$
Relation: $\bar{a}_{\gamma\kappa} = a_{\alpha\beta} \dfrac{\partial u^\alpha}{\partial \bar{u}^\gamma} \dfrac{\partial u^\beta}{\partial \bar{u}^\kappa}$; $a_{\sigma\tau} = \bar{a}_{\gamma\kappa} \dfrac{\partial \bar{u}^\gamma}{\partial u^\sigma} \dfrac{\partial \bar{u}^\kappa}{\partial u^\tau}$

eg. $g_{\alpha\beta} = \underbrace{\vec{x}_\alpha}_{\frac{\partial x}{\partial u^\alpha}} \cdot \underbrace{\vec{x}_\beta}_{\frac{\partial x}{\partial u^\beta}}$ are components of a cov tensor of 2nd order.

Def Mixed tensor $a_\sigma{}^\tau = \bar{a}_\nu{}^k \frac{\partial \bar{u}^\nu}{\partial u^\sigma} \frac{\partial u^\tau}{\partial \bar{u}^k}$; $\bar{a}_\nu{}^k = a_\alpha{}^\beta \frac{\partial u^\alpha}{\partial \bar{u}^\nu} \frac{\partial \bar{u}^k}{\partial u^\beta}$

Remarks : Consider invariant $L = a_\alpha{}^\beta b^\alpha c_\beta$.

Def Most general case : $L = h_{\alpha_1 \cdots \alpha_r}{}^{\beta_1 \cdots \beta_s} a^{\alpha_1} \cdots a^{\alpha_r} b_{\beta_1} \cdots b_{\beta_s}$

where L : invariant, a^{α_r} contra. b_{β_s} covariant vectors. $h_{\cdots}^{\cdots} \in \mathbb{R}^{(n^{r+s})}$.

Relation : $\bar{h}_{\gamma_1 \cdots \gamma_r}{}^{k_1 \cdots k_s} = h_{\alpha_1 \cdots \alpha_r}{}^{\beta_1 \cdots \beta_s} \frac{\partial u^{\alpha_1}}{\partial \bar{u}^{\gamma_1}} \cdots \frac{\partial u^{\alpha_r}}{\partial \bar{u}^{\gamma_r}} \frac{\partial \bar{u}^{k_1}}{\partial u^{\beta_1}} \cdots \frac{\partial \bar{u}^{k_s}}{\partial u^{\beta_s}}$.

Remark 1. Physical examples
- 0-order temperature field.
- 1- ·· force field.
- 2- stress of elastic body

2. if tensor $h_\alpha{}^{\cdots\beta}$ is defined in one u^α, then $\bar{h}$ will be defined w.r.t other coordinate systems.

3. (Any?) vector field in Euclidean space (plane) is 1st-order tensor

Basic Rules of Tensor Calculus

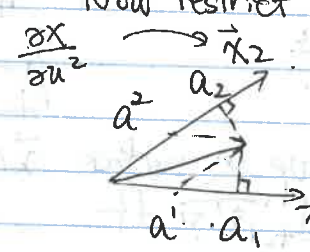
- Tensor of the same type → vector space.
  1. "+" 2. 数乘
- ~~product~~ / outer product : every component × every component.
  $h_{\alpha_1 \cdots \alpha_p \gamma_1 \cdots \gamma_r}{}^{\beta_1 \cdots \beta_q k_1 \cdots k_s} = a_{\alpha_1 \cdots \alpha_p}{}^{\beta_1 \cdots \beta_q} b_{\gamma_1 \cdots \gamma_r}{}^{k_1 \cdots k_s}$.

??? form measurement.

- Contraction $\alpha = \beta$ → contra ↓, cov ↓.
  $a_\alpha{}^\alpha = a_1{}^1 + a_2{}^2 \cdots a_n{}^n$ → scalar.
  Proof $\bar{a}_\nu{}^\nu = a_\alpha{}^\beta \frac{\partial u^\alpha}{\partial \bar{u}^\nu} \frac{\partial \bar{u}^\nu}{\partial u^\beta} = a_\alpha{}^\beta \delta_\alpha{}^\beta = a_\alpha{}^\alpha$.
- Inner Product : contraction is appied to 2 ~~produ~~ tensors w.r.t indices of diff factors

Vectors in a Surface. Contravariant Metric Tensor.

Now restrict $n = 2$.



Theorem Contravariant component a^α in a surface at P are the lengths of the parrellel projection of v. in the tangent space E(P), with unit vector. $\vec{x}_1, \vec{x}_2$ whose length units are $\sqrt{g_{11}}, \sqrt{g_{22}}$.

Theorem Covariant ··· a_α are orthogonal projections of v. Length units are $1/\sqrt{g_{\alpha\alpha}}$.

Remark : 1. a^α, a_α can be converted as

$$\underline{a_\alpha} = x_\alpha \cdot v = x_\alpha \cdot a^\beta x_\beta = \underline{g_{\alpha\beta} a^\beta}$$

$a^\beta = g^{\alpha\beta} a_\alpha$; $g^{11} = \frac{g_{22}}{g}$, $g^{12} = \frac{-g_{12}}{g}$, $g^{22} = \frac{g_{11}}{g}$.

$\boxed{\cancel{\text{Def}}}$ 2 Conjugate: $a_{\alpha\beta} b^{\beta\gamma} = \delta_\alpha^\gamma \begin{cases} 1 & \alpha=\gamma \\ 0 & \alpha\neq\gamma \end{cases}$

eg. $g_{\alpha\beta}$, $g^{\alpha\beta}$ are conjugate.

contra basis vector → 长什么样?

3. As $E(P)$ is spanned by $x_\alpha \equiv \frac{\partial x}{\partial u^\alpha}$, it's equally spanned by the contravariant basis vector $x^\alpha = g^{\alpha\beta} x_\beta$.

x^α, x_α are conjugate, $x_\alpha \cdot x^\beta = \delta_\alpha^\beta$.

4. If coordinates are Cartesian, the $g_{\alpha\beta} = \delta_{\alpha\beta}$, $a^\alpha = a_\alpha$.

5 Transition between cov and contra components is accomplished by inner product of the component and the metric tensor.

$a_\alpha = a^\beta \cdot g_{\alpha\beta}$.

Special Tensors

δ tensor — similar to δ ~~just in 1st-order~~

Proof $\bar{a}_\rho^\sigma = \delta_\alpha^\beta \frac{\partial u^\alpha}{\partial \bar{u}^\rho} \frac{\partial \bar{u}^\sigma}{\partial u^\beta} = \delta_\rho^\sigma = \bar{\delta}_\rho^\sigma$

ϵ - tensor $\triangleq$ $\epsilon_{11}=0$ $\epsilon_{12}=\sqrt{g}$ $\epsilon_{21}=-\sqrt{g}$. $\epsilon_{22}=0$.

相当于叉乘.

Proof: The def in any coord undergoes ~~contra~~ rules.

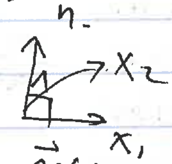
$\triangleq$ $\epsilon^{11}=0$ $\epsilon^{12}=1/\sqrt{g}$ $\epsilon^{21}=-1/\sqrt{g}$ $\epsilon^{22}=0$.

Remarks: 1. Skew-Symmetric.

2. $\epsilon^{\alpha\gamma}\epsilon_{\gamma\beta} = -\delta_\beta^\alpha$; $\epsilon^{\alpha\gamma}\epsilon_{\beta\gamma} = \delta_\beta^\alpha$.

(proof: $\epsilon^{\alpha\gamma}\epsilon_{\beta\gamma} = \sum_{\gamma=1}^{2} \epsilon^{\alpha\gamma}\epsilon_{\beta\gamma} = \begin{cases} 0 & (\alpha\neq\beta) \leftarrow \cancel{项} \times 0 \\ 1 & (\alpha=\beta) \end{cases}$

Normal to a Surface. $\boxed{\text{Def}}$ $\vec{n} = \frac{x_1 \times x_2}{|x_1 \times x_2|} = \frac{x_1 \times x_2}{\sqrt{g}}$.



Measurement of Lengths and Angles in a Surface

$\boxed{\text{Def}}$ (Length). A curve $u^\alpha = u^\alpha(t)$ be a curve on surface $\vec{x}(u^1, u^2)$.

$s = \int^{s_1} ds = \cancel{\int \sqrt{\dot{x}\cdot\dot{x}}}\ \int_{s_1}^{s_2} \sqrt{dx \cdot dx} = \int_{t_0}^{t_1} \sqrt{\vec{x}' \cdot \vec{x}'}\, dt$

$= \int_{t_0}^{t_1} \sqrt{\vec{x}_\alpha u^{\alpha'} \cdot \vec{x}_\beta u^{\beta'}}\, dt \triangleq \int_{t_0}^{t_1} \sqrt{g_{\alpha\beta} u^{\alpha'} u^{\beta'}}\, dt$.

where $u^{\alpha'} = \frac{du^\alpha}{dt}$.

$(u^{1'}\ u^{2'}) \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} u^{1'} \\ u^{2'} \end{pmatrix}$

$s'^2 = x'^2 \cdot x'^2$.

$ds^2 = dx \cdot dx$

$g_{\alpha\beta}$
$= \vec{x}_\alpha \cdot \vec{x}_\beta$.
depends on u, v
$g_{\alpha\beta}$: component ~~under~~ w.r.t. u, v.

Angle: $\vec{a} = a^\alpha \vec{x}_\alpha \quad \vec{b} = b^\beta \vec{x}_\beta$

$$\vec{a} \cdot \vec{b} = a^\alpha \vec{x}_\alpha \cdot b^\beta \vec{x}_\beta = g_{\alpha\beta} a^\alpha b^\beta$$
$$= g_{\alpha\beta} g^{\alpha\nu} a_\nu b^\beta = \delta_\beta^\nu a_\nu b^\beta = a_\beta b^\beta = g^{\alpha\beta} a_\beta b_\alpha = g^{\alpha\beta} a_\alpha b_\beta$$

$$|a| = \sqrt{g^{\alpha\beta} a_\alpha a_\beta}$$

$$\cos\gamma = \frac{a \cdot b}{|a||b|} = \frac{g_{\alpha\beta} a^\alpha b^\beta}{\sqrt{g_{\mu\nu} a^\mu a^\nu}\sqrt{g_{\sigma\tau} b^\sigma b^\tau}} = \frac{g^{\alpha\beta} a_\alpha b_\beta}{\sqrt{g^{\mu\nu} a_\mu a_\nu}\sqrt{g^{\sigma\tau} b_\sigma b_\tau}}$$

$\boxed{\text{Theorem}}$ coordinate ~~on~~ a surface is orthogonal iff $g_{12} = 0 \ \forall P$.

$$\vec{x}_1 \cdot \vec{x}_2 = g_{12} = 0$$

$$\sin\gamma = \sqrt{1 - \cos^2\gamma} = \frac{|a_1 b_2 - a_2 b_1|}{\sqrt{g}\sqrt{g^{\alpha\beta} a_\alpha a_\beta}\sqrt{g^{\mu\nu} b_\mu b_\nu}}$$

Area $\boxed{\text{Def}}$ $A(H) = \iint \sqrt{g}\, du^1 du^2$

$$dA = \sqrt{g}\, du^1 du^2$$

Remarks 1. parrallelogram $\square\ \vec{x}_1 du^1 \times \vec{x}_2 du^2$ on the surface

$$= |x_1 du^1 \times x_2 du^2| = |\vec{x}_1 \times \vec{x}_2|\, du^1 du^2 = \sqrt{g}\, du^1 du^2.$$

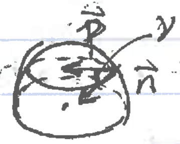
$$(x_1 \times x_2)^2 = (x_1 \cdot x_1)(x_2 \cdot x_2) - (x_1 \cdot x_2)^2 = g.$$

2. $g_{\alpha\beta}$ of the 1st-FF $\Rightarrow$ measure lengths, areas, angles ----
$\Rightarrow$ metric in a surface is determined.

SECOND-FUNDAMENTAL FORM: GAUSSIAN AND MEAN CURVATURE OF A SURFACE

2nd-FF

Geometric shape of a surface in the neighborhoods of any points



Start from the curvature of a curve on a surface.

surface $S: \vec{x}(u^1 u^2)$. curve $C \subset S: u^1(s), u^2(s)$

$$\cos\gamma = \vec{p}\cdot\vec{n} = \ddot{\vec{x}}/\kappa \cdot \vec{n}$$

$$\cancel{x = x_\alpha u^\alpha} \qquad \dot{x} = \frac{\partial x}{\partial u^1}\frac{\partial u^1}{\partial s} + \frac{\partial x}{\partial u^2}\frac{\partial u^2}{\partial s} = x_\alpha \dot{u}^\alpha$$

$$\ddot{x} = \cancel{x_\alpha \dot{u}^\alpha + x_\alpha \ddot{u}^\alpha}\ x_{\alpha\beta}\dot{u}^\alpha\dot{u}^\beta$$

$$= x_{\alpha\beta}\dot{u}^\alpha\dot{u}^\beta + x_\alpha\ddot{u}^\alpha.$$

① $b_{\alpha\beta} = \vec{x}_{\alpha\beta}\cdot\vec{n}$

② $= -\vec{x}_\alpha\cdot\vec{n}_\beta$

③ $= \frac{1}{\sqrt{g}}|\vec{x}_1\, \vec{x}_2\, \vec{x}_{\alpha\beta}|$

Def: ∵ $x_\alpha \perp \vec{n} \Rightarrow \ddot{x}\cdot\vec{n} = (\vec{x}_{\alpha\beta}\cdot\vec{n})\dot{u}^\alpha\dot{u}^\beta$

Denote $\boxed{b_{\alpha\beta} = \vec{x}_{\alpha\beta}\cdot\vec{n} = -\vec{x}_\alpha\cdot\vec{n}_\beta}$, then $\boxed{b_{\alpha\beta}\,du^\alpha du^\beta \Rightarrow \text{2nd-FF}}$

$$= b_{11}du^1du^1 + 2b_{12}du^1du^2 + b_{22}(du^2)^2.$$

Remarks:

1. Proof: $\vec{x}_\alpha\cdot\vec{n} = 0 \Rightarrow \cancel{\vec{x}_{\alpha\beta}\cdot\vec{n}_\beta = 0}$

$$\Rightarrow \vec{x}_{\alpha\beta}\cdot\vec{n} + \vec{x}_\alpha\cdot\vec{n}_\beta = 0.$$

$$\therefore b_{\alpha\beta} = -\vec{x}_\alpha\cdot\vec{n}_\beta.$$

Independent of curve C on S.

2. $b_{\alpha\beta}du^1du^2 = -d\vec{x}\cdot d\vec{n}$ is invariant.

3. $b_{\alpha\beta} = \vec{x}_{\alpha\beta}\cdot\vec{n} = \vec{x}_{\alpha\beta}\cdot\left(\frac{|\vec{x}_1\times\vec{x}_2|}{\sqrt{g}}\right) = \frac{1}{\sqrt{g}}|\vec{x}_1\ \vec{x}_2\ \vec{x}_{\alpha\beta}|$

Arbitrary & Normal Sections of a Surface.

$$\kappa\cos\gamma = \ddot{x}\cdot\vec{n} = \cancel{x_{\alpha\beta}}\, b_{\alpha\beta}\dot{u}^\alpha\dot{u}^\beta \qquad \left(\dot{u}^\alpha = \frac{du^\alpha}{dt}\frac{dt}{ds}\right) = \frac{u^{\alpha\prime}}{s'}$$

$$= \frac{b_{\alpha\beta}u^{\alpha\prime}u^{\beta\prime}}{(s')^2} = \frac{b_{\alpha\beta}u^{\alpha\prime}u^{\beta\prime}}{g_{\alpha\beta}u^{\alpha\prime}u^{\beta\prime}} = \frac{b_{\alpha\beta}du^\alpha du^\beta}{g_{\alpha\beta}du^\alpha du^\beta}.$$

$(s')^2 = g_{\alpha\beta}u^{\alpha\prime}u^{\beta\prime}$

where κ: curvature of C.

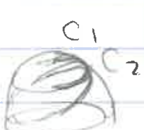
γ: angle between principle normal of C, & normal to S.

Note: $du^\alpha \hat{=} u^{\alpha\prime} \neq \dot{u}^\alpha$

Geometric Interpretation: κ depends of (principle curvature $\vec{p}$ } osculating plane; $\dot{u}^1 : \dot{u}^2 \Rightarrow \vec{t}$

NOW. CONSIDER ANY NEIGHBOR

Theorem: All curves $(r \geq 2)$ which pass through any fixed $\dot{x} = x_\alpha\dot{u}^\alpha$ point P and have at P the same osculating plane, also have the same curvature.



推: can restrict on plane curves on S without loss of generalization.

Def: Normal Sections of S: curves ~~that~~ of ~~intersection~~ of S, and the plane ~~that~~ which pass through $\vec{t}$ and $\vec{n}$; $\Leftrightarrow \gamma = 0$ or π. $\vec{n} \in$ osculating plane

If tangent is fixed, $\kappa\cos\gamma = \kappa_n$ ← constant

$\gamma = 0 \Rightarrow \kappa = \kappa_n.$

$\gamma = \pi \Rightarrow \kappa = -\kappa_n$

[Def] Normal Curvature k_n.

Note k_n depends only on direction of $\vec{t}$ ⇒ k_n normal curvature

$\vec{k}_n = k_n \vec{n}$ ⇒ normal curvature vector ... $= \dfrac{b_{\alpha\beta} du^\alpha du^\beta}{g_{\alpha\beta} du^\alpha du^\beta}$

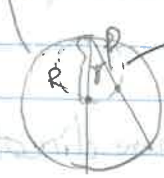
normal section: plane through $\vec{t}, \vec{n}$ only depends on $\vec{t}$.

its curvature: k_n ←, or when $\vec{p}t = 0 / \vec{t}$.

other curves that share the same $\vec{t}$: $k\cos\gamma = k_n$

that is: on a circle.

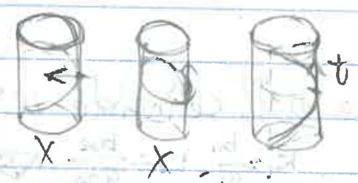
[Theorem] Meusnier : The center of curvature of all curves on a surface S which pass through an arbitrary fixed point P and $\vec{t}(P)$ has the same direction, lie on a circle of K of Radius $\frac{1}{2}|R|$ which lies in the normal plane and has ≥ 1 order of contact.



ρ lies on dotted circle.

⇒ We can restrict our consideration to normal section of S

Asymptotic Direction : $k_n = 0 = \dfrac{b_{\alpha\beta} du^\alpha du^\beta}{g_{\alpha\beta} du^\alpha du^\beta}$ ← 2nd FF vanish.



$k_n = 0,\ k > 0$.

← $\vec{p}$ ⇒ $\gamma = 90°$ ⇒ $\vec{p} \subset$ Tangent (P).

⇒ $\vec{b} \parallel \vec{n}$ ($\vec{b} = \vec{n}$).

Osculating plane = Tangent (P).

?? [Theorem] Asymptotic curves iff $b_{11} = 0$ $b_{22} = 0$.

Proof $k_n = 0 \Leftrightarrow b_{\alpha\beta} du^\alpha du^\beta = 0$

$b_{11}(du^1)^2 + b_{22}(du^2)^2 + \underbrace{2 b_{12} du^1 du^2}_{??} = 0$.

Elliptic Parabolic hyperbolic.

Till now, any curve → plane curves $\xrightarrow{\text{Meusnier}}$ normal section.

↳ ⊥ tangent plane.

curvature of a normal section : $k_n = \dfrac{b_{\alpha\beta} du^\alpha du^\beta}{g_{\alpha\beta} du^\alpha du^\beta}$

(/normal curvature)

consider the sign of k_n.

$b = \det(b_{\alpha\beta}) = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = b_{11} b_{22} - (b_{12})^2$

$b > 0$. elliptic

$b = 0$ parabolic.

$b < 0$

$b>0$ elliptic : center of curvature lie on the same side

$b=0$ parabolic : exactly 1 direction $k_n=0$

$b<0$ hyperbolic / saddle point 2 direction $k_n=0$

$\boxed{\text{Theorem}}$ 2nd-FF approximate how much the surface deviate the ~~pt~~ tangent plane, quantitatively: $\frac{1}{2} b_{\alpha\beta} du^\alpha du^\beta$

Proof : Taylor expand $Q = \vec{x}(u^1+h^1, u^2+h^2) + o((h^1+h^2)^2)$.

$$Q = x(u^1,u^2) + h^\alpha x^\alpha + \frac{1}{2} h^\alpha h^\beta x_{\alpha\beta}$$

Consider the distance of Q from $E(P)$.

$$(Q-P)\cdot\vec{n} = (h^\alpha x^\alpha + \frac{1}{2} h^\alpha h^\beta x_{\alpha\beta})\,\vec{n}$$

$$= \frac{1}{2} h^\alpha h^\beta (x_{\alpha\beta}\cdot\vec{n}) = \frac{1}{2} b_{\alpha\beta} h^\alpha h^\beta$$

Set $h^\alpha = du^\alpha$.

Principle curvature.

$\boxed{\text{Def}}$ Line of curvature : A curve of S where direction at every point is a principal direction.

Lines of curvature

$\boxed{\text{Theorem}}$ $k_1 \perp k_2$.

Gaussian & Mean Curvature

$\boxed{\text{Theorem}}$ The coordinate curves $(u^1 u^2)$ coincide with lines of curve iff $g_{12}=0$, $b_{12}=0$, $k_1 = \frac{b_{11}}{g_{11}}$ $k_2 = \frac{b_{22}}{g_{22}}$ $\cancel{k}\ b_{11}$

? $b_{\alpha\beta} g^{\alpha\beta} = b_\alpha^\alpha$??

Analytic solution is $k_n^2 - b_{\alpha\beta} g^{\alpha\beta} k_n + \frac{b}{g} = 0$.

$$K = k_1 k_2 = b/g$$

$$H = \frac{1}{2}(k_1+k_2) = \frac{1}{2} b_{\alpha\beta} g^{\alpha\beta} \overset{?}{=} \frac{1}{2} b_\alpha^\alpha$$ when $u^1 u^2$ is line of curve?

K, H : invariant.

K : only depends on 1st-FF, not 2nd-FF

Euler's theorem

Dupin's indicatrix

$\boxed{\text{Theorem}}$ k_n can be represented ~~in~~ in terms of k_1, k_2.

k_2 k_n k_1 α

$$k_n = k_1 \cos^2\alpha + k_2 \sin^2\alpha$$

Proof : cf P132.

Remarks: Euler + Meusnier theorem $\Rightarrow$ complete info on curvature of any curve on surface.

$\boxed{\text{Def}}$ Dupin indicatrix (双曲线)

$$k_1 x_1^2 + k_2 x_2^2 = \pm 1$$

Remarks 1: The Dupin indictrix is more than algebraic incident.
Geometric meaning: Intersection of S and plane // to $E(P)$.

intersection: $\frac{1}{2} b_{\alpha\beta}\, du^\alpha du^\beta = \pm \varepsilon$.

choose u^α $\Rightarrow$ $b_{11}(du^1)^2 + b_{22}(du^2)^2 = \pm 2\varepsilon$.

$\Leftrightarrow$ $k_1 g_{11}(du^1)^2 + k_2 g_{22}(du^2)^2 = \pm 2\varepsilon$

$\Rightarrow$ intersection is approximately a conic section which is similar and similarly(相似,)cs) placed to the Dupin indicatrix at P.

$\boxed{\text{Def}}$ Flat ~~pla~~ point: $b_{\alpha\beta} \equiv 0 \,|_{P_0}$

Flat Points, Saddle points of higher type

Analogous to formulae of Frenet, which describes $\dot{t}, \dot{p}, \dot{b}$ in terms of t, p, b: $\begin{pmatrix} \dot{t} \\ \dot{p} \\ \dot{b} \end{pmatrix} = \begin{pmatrix} & \kappa & \\ -\kappa & & \tau \\ & -\tau & \end{pmatrix}\begin{pmatrix} t \\ p \\ b \end{pmatrix}$. Here we investigate the local coordinat $\{x_1, x_2, n\}$ and how they chang $\{$~~dx_α, dn~~ $x_{\alpha\beta}, n_\alpha\}$ in terms of themslf

Formulae of Weingarten and Gauss

1. n_α: since $\vec{n}\cdot\vec{n} = 1$, $2\vec{n}\cdot n_\alpha = 0$ $\Rightarrow$ $n_1, n_2 \in E(P)$.

$n_\alpha = C_\alpha^\gamma x_\gamma$, determine C_α^γ.

? $b_\alpha^\beta \cdot g_\alpha^\beta$. mixed tensor 転換?

$-b_{\alpha\sigma} = n_\alpha \cdot x_\sigma = C_\alpha^\gamma x_\gamma \cdot x_\sigma = C_\alpha^\gamma g_{\gamma\sigma}$.

同 $\times g^{\sigma\tau}$

$\underline{-b_{\alpha\sigma} g^{\sigma\tau}} \stackrel{?}{=} \underline{-b_\alpha^\tau} = C_\alpha^\gamma g_{\gamma\sigma}\cdot g^{\sigma\tau} = C_\alpha^\gamma \delta_\gamma^\tau = \underline{C_\alpha^\tau}$.

$g_{\alpha\beta}$ 5 δ_γ^τ?

$g_{\gamma\sigma}g^{\sigma\tau} = \delta_\gamma^\tau$?

$$\therefore \boxed{\vec{n}_\alpha \equiv \frac{\partial \vec{n}}{\partial u^\alpha} = -b_\alpha^\beta \vec{x}_\beta \quad (b_\alpha^\beta = g^{\sigma\beta} b_{\sigma\alpha}}$$

Remarks:

1. when u^α are line of curve ($b_{12} = g_{12} = 0$):
$\vec{n}_\alpha = -\frac{b_{\alpha\alpha}}{g_{\alpha\alpha}} x_\alpha \Leftrightarrow \vec{n}_\alpha = -k_\alpha x_\alpha \Leftrightarrow \boxed{\vec{x}_\alpha + R_\alpha \vec{n}_\alpha = 0}$

2. for displacement in principle direction: $\kappa_n dx + dn = 0$

De

2. $\Gamma_{\alpha\beta}^{\gamma}$ $\quad x_{\alpha\beta} = \Gamma_{\alpha\beta}^{\gamma} x_{\gamma} + a_{\alpha\beta}\vec{n} \quad \times x^{\lambda}$

$x_{\alpha\beta} \cdot x_{\lambda} = \Gamma_{\alpha\beta}^{\gamma} x_{\gamma} \cdot x_{\lambda} = \Gamma_{\alpha\beta}^{\gamma} g_{\gamma\lambda} \quad \times g^{\lambda k}$

$x_{\alpha\beta} \cdot x_{\lambda} g^{\lambda k} = \Gamma_{\alpha\beta}^{\gamma} \delta_{\gamma}^{k}$ (when $\gamma = k$).

$\Gamma_{\alpha\beta}^{\gamma} = \underbrace{x_{\alpha\beta} \cdot x_{\lambda}} g^{\lambda k}$.

$\boxed{\text{Def}}$ 记 $\Gamma_{\alpha\beta\lambda} \equiv x_{\alpha\beta} \cdot x_{\lambda}$.

Christoffel $\quad \Gamma_{\alpha\beta}^{k} = g^{\lambda k} \Gamma_{\alpha\beta\lambda}$

symbol

Remarks: 1. symmetric about $\alpha\beta$.

2. is not tensor.

3. write in terms of g, u;

$$\Gamma_{\alpha\beta\lambda} = \frac{1}{2}\left[\frac{\partial g_{\beta\lambda}}{\partial u^{\alpha}} + \frac{\partial g_{\alpha\lambda}}{\partial u^{\beta}} - \frac{\partial g_{\alpha\beta}}{\partial u^{\lambda}}\right]$$

Intergrability conditions of the formulae of Weingarten and Gauss

Formulae of Frenet: $\forall\, k, \tau \Rightarrow \exists\, C$, s.t. $k(C) = k$, $\tau(C) = \tau$

(the ODE always have solution).

" of Weingarten & Gauss: Not the case

unless some integrability conditions

$\boxed{\text{Theorem}}$ (Fundamental)(Egregium). Gaussian curvature K is independent of the 2nd-FF, only on 1st F.F (and their 1st, 2nd derivatives). [see P145).

Remarks: 1. important in connexion with bending & isometric mapping.

GEODESIC CURVATURE AND GEODESICS.

Geodesic Curvature.

Shortest line on plane: straight $\Rightarrow$ curvature vanishes
~~Curve~~ " " on surface: $\Rightarrow$ geodesic curvature k_g vanish $\Downarrow$ $k_g \equiv 0$ $\Downarrow$ geodesic curve

1. Sec 52. minimum length necessarily is a geodesic.

(relation to calculus of variation). 2. sufficient conditions in order that C be the shortest paths

$\boxed{\text{Def.}}$ Geodesic Curvature k_g: curvature of C' which is C projected on E(P), direction ~~of~~ $\vec e = \vec n \times \vec t$

Algebraically: $|k_g| = k\sin\gamma$. $\leftarrow$ geodesic

$k_n = k\cos\gamma$ $\leftarrow$ normal curvature

Remarks: 1. $\vec k = \vec k_n + \vec k_g = k_n \vec n + k_g \vec e$

2. $k_g = |\dot x\ \ddot x\ n|$

3. Depends on both curve and its surface

4. $k_n \to g_{\alpha\beta};\ b_{\alpha\beta}$

$\boxed{\text{Theorem}}$ $\to$ $k_g \to g_{\alpha\beta}$ ~~+ $b_{\alpha\beta}$~~ only on 1st-FF

Proof; $k_g = (\dot x \times \ddot x)\cdot \vec n$

$\dot x = x_\alpha \dot u^\alpha$

$\ddot x = x_{\beta\gamma}\dot u^\gamma \dot u^\beta + x_\epsilon \ddot u^\epsilon$

$= x_\epsilon \ddot u^\epsilon + [\Gamma_{\beta\gamma}^{\epsilon} x_\epsilon + b_{\beta\gamma}\vec n]\dot u^\gamma \dot u^\beta.$

又 $x_1 \times x_2 = \sqrt{g}\,\vec n$, $x_\alpha \times x_\alpha = 0$. $n\cdot n = 1$

代入展开，消 $b_{\alpha\beta}$.

Geodesics. $\boxed{\text{Def}}$ $k_g \equiv 0 :\to$ C is geodesic

$\boxed{\text{Theorem}}$ 1. Straight line on any surface is geodesic.

2. Curve C is geodesic $\Leftrightarrow$ $\vec n \in$ Osculating Plane.

Proof $k_g = k\sin\gamma \Rightarrow k = 0$ or $\gamma = 0/2\pi$.

$(\vec p = \vec n)$.

Arcs of min length.

if C_1C_2 is min length $\Rightarrow$ C_1C_2 is an arc of geodesic.

Remark: 1. Prove by variation method.

2. Not the inverse, eg.

? variational method ......

Geodesic parrellel coordinates

[Def] A field of Geodesics — generation of orthogonal parrellel cord on plane.

1-param family of geodesic on a surface, if through every point of S, ~~#~~ $\exists$ passes exactly one of those geodesic.

e.g. /// parallel straight line.

sphere is not.

Geodesic parrell coordinate: coord system $u^{1*} u^{2*}$. We chose a field of geodesic on S ~~and~~ and set $u^{*2} = \text{const}$. u^{1*} is the orthogonal direction.

Under this coord system, ~~there~~ are some good properties.

$g_{12}^{*} = 0$ (orthogonal).

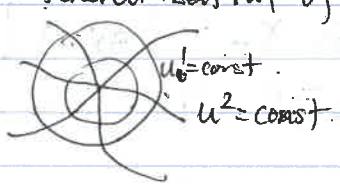
$$k_g|_{u^{2*}=\text{cost}} = 0 \Rightarrow \frac{\partial g_{11}^{*}}{\partial u^{2*}} = 0.$$

$$K = -\frac{1}{\sqrt{g}} \frac{\partial^2 \sqrt{g}}{\partial (u^1)^2}.$$

[Theorem] Sufficient condition for G be an arc ~~of~~ min length: G can be embeded in a field of geodesic.

Geodesic Polar Coordinate

Generalization of polar coord on plane.

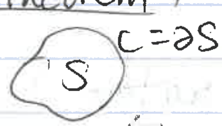


$u^2 = \text{const}$: geodesic.

$u^1 = \text{const}$ radius circle.

Theorem of Gauss-Bonnet Integral Curvature

[Theorem] Gauss-Bonnet.

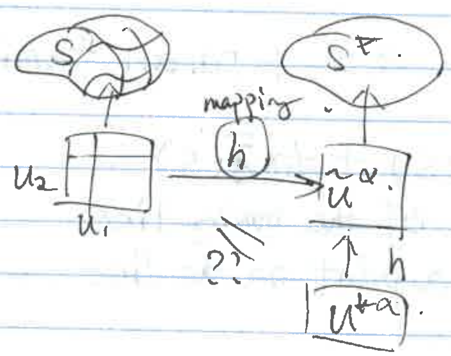


$$\int_C k_g\, ds + \underbrace{\iint_S K\, dA}_{\text{integral curvature}} = 2\pi.$$

MAPPING

Preliminaries

Isometric. Conformal Equiareal.



S, S^* coord sys are the same ?? have corsp

Isometric : length
Conformal angle
Equiareal area.

Isometric mapping. Bending, Intrinsic Geometry of a surface.

$\boxed{\text{Theorem}}$. S, S^* are isometric iff. $g_{\alpha\beta} = g^*_{\alpha\beta}$ at corresp points.

Proof : length of curve C $u^\alpha = h^\alpha(t)$:

$$S(t_0) = \int_0^{t_0} \sqrt{g_{\alpha\beta} h^{\alpha\prime} h^{\beta\prime}}\, dt. \quad \left(h^{\alpha\prime} = \frac{dh^\alpha}{dt} \right).$$

$\boxed{\text{Theorem}}$ Isometric surface have the same Gaussian curvature K at each corrsp point. Corrsp curves have the same K_g (geodesic curvature) at corsp point.

Proof : K only depends on $g_{\alpha\beta}$.

Def Bending : A continuous deformation that preserves length.

Remarks : 1. ~ applicable Surfaces can be transformed by bending.

2. Bending Invariants, eg. properties that ~~depend only on~~ $g_{\alpha\beta}$.

Def Intrinsic / Absolute properties : those properties depend on the 1st-FF and independant of 2nd-FF.

Embedding & Metric tensor :

?? eg ?

1. obtain the $g_{\alpha\beta}$ by metric of the embedding Euclidean space.
   $\Rightarrow$ i.e. the metric of the space ($g_{\alpha\beta}$) induce a metric on surface.
2. if the analytic form of length, area, angle $g_{\alpha\beta}$ have been derived,
   $\Leftrightarrow$ metric tensor being given $\Rightarrow$ embedding becomes unessential.

Ruled Surface. Developable Surface

Goal : find surfaces isometric to a plane.
$\Leftrightarrow$ developable

$\boxed{\text{Def}}$ Ruled Surfaces : contains ≥ 1 1-param family of straight lines. lines : generator

$$\vec{x}(s,t) = \vec{y}(s) + t\vec{\gamma}(s)$$

where $\vec{\gamma}$: unit vector of the moving line

$\vec{y}(s)$: traj of a point on the line.

eg. tangent surfaces

$$\vec{x}(s,t) = y(s) + t\,\vec{t}(s).$$

$$\cdots\ \vec{p}(s)$$

$$\cdots\ b(s).$$

choose $\vec{\gamma}$ as $\vec{t}/\vec{p}/\vec{b}$

?? FF $g_{\alpha\beta}$? $ds^2 = dx \cdot dx$?

$\boxed{\text{Def}}$ Developable surface. $x(s,t) = y(s) + t\gamma(s)$.

iff $|\dot{y}\ \gamma\ \dot{\gamma}| = 0$.

Spherical image of a surface, 3rd-FF

Isometric mapping of dev'able surface.

$\boxed{\text{Def}}$ Gaussian Spherical mapping. $\vec{n}$

$\boxed{\text{Def}}$ 3rd-FF : the 1st-FF of spherical image of S.

$$ds^{*2} = C_{\alpha\beta}du^\alpha du^\beta \qquad C_{\alpha\beta} = \vec{n}_\alpha \cdot \vec{n}_\beta \ ??$$

$$\begin{cases} \mathrm{I} := ds^2 = dx \cdot dx \\ \mathrm{II} := -dx \cdot dn. \\ \mathrm{III} := dn \cdot dn = ds^{*2}. \end{cases}$$

Remark: 1. $C_{\alpha\beta}$ can be represented as $b_{\alpha\beta}$. $g_{\alpha\beta}$.

$$C_{\alpha\beta} = \vec{n}_\alpha \cdot \vec{n}_\beta = b_\alpha^{\ \gamma}\vec{x}_\gamma \cdot b_\beta^{\ \eta}\vec{x}_\eta = b_\alpha^{\ \gamma} b_\beta^{\ \eta} g_{\gamma\eta} = b_{\alpha\kappa} b_{\beta\gamma} g^{\kappa\gamma}.$$

$$C = \begin{vmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{vmatrix} = \frac{b^2}{g}$$

2. $C_{\alpha\beta} = 2H b_{\alpha\beta} - K g_{\alpha\beta}$ $\quad$ H : mean. curv

K : Gaussian curv.

Proof : by calculation, choose coord sys as line of curvature.

direct of principle curv.

$\Leftrightarrow g_{12} = b_{12} = 0.$

$$\cancel{K} = K_n^2 - b_{\alpha\beta}g^{\alpha\beta}K_n + \frac{b}{g} = 0$$

the. $K = K_1 K_2 = \frac{b}{g}$ $\quad$ $H = \frac{1}{2}(K_1 + K_2) = \frac{1}{2} b_\alpha^{\ \alpha}$.

代入.

$\boxed{\text{Theorem}}$ Rodrigues : $\frac{dA^*}{dA} = |K| \therefore \frac{\sqrt{c}}{\sqrt{g}} = \frac{|b|}{\sqrt{g}}\frac{1}{\sqrt{g}} = |K|$.

$K = \frac{\text{the element area of } S^*}{\text{〃 〃 on } S}$.

~~todo~~ :

Remarks : Developable surface $S \Rightarrow \vec{n}$ is same for all points on the generating straight line $\Rightarrow$ S^* is a curve.

$\boxed{\text{Theorem}}$. A (suff small) portion of S of class $r \geq 3$ can be mapped isometrically into a plane iff it's developable.

Conformal Map.

Angle of curves : $\langle \vec{t}, \vec{t}^* \rangle, \cos\vartheta = \frac{v \cdot v^*}{|v| \cdot |v^*|} = \frac{g_{\alpha\beta} h^{\alpha'} h^{\beta'}}{\sqrt{g_{\kappa\lambda} h^{\kappa'} h^{\lambda'}} \sqrt{g_{\sigma\tau} h^{\sigma'} h^{\tau'}}}$

$v = \frac{d}{dt}\{x(h^1(t), h^2(t))\} = x_\alpha h^{\alpha'}$

$\boxed{\text{Theorem}}$ ~~A~~ Mapping of S, S^* are conformal iff, $g_{\alpha\beta}$ $g_{\alpha\beta}^*$ are proportional:

$$g_{\alpha\beta}^* = \eta(u^1, u^2)\, g_{\alpha\beta} \cdot \quad \eta > 0.$$

Note : η depends on P, but not on curve direction

Conformal Mapping of Surfaces into a Plane $\boxed{\text{Def}}$

$S \xrightarrow{I} E$ plane.

ds. $\quad u^1 u^2$.

$$ds^2 = \eta(u^1, u^2)[(du^1)^2 + (du^2)^2]$$

then u^1, u^2 : isothermic coord on S.

Map of $E_1\ E_2$.

$$w \equiv u^{1*} + i u^{2*} = h(u), \quad u = u^1 + i u^2.$$

then mapping h is conformal : complex theory ??

if h is regular.

$\boxed{\text{Theorem}}$ Any simply-connect portion of S can be conformally mapped into a plane.

TODO : proof ....

Isotropic Curves and Isothermic Coordinates

Embed real Euclidean space to complex

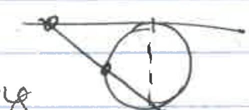
Real Curve $\vec{x}(t)$, $t \in \mathbb{R}$.
Complex Curve $\vec{x}(t)$ $t \in \mathbb{C}$, $t = t_1 + i t_2$.

$\boxed{\text{Def}}$ Isotropic / Minimal Curve : $ds^2 = 0$.
Any arc between 2 points is 0.

---- many theory involved complex analysis / diff. Eq ---

Conformal Mapping of a Sphere into a Plane.

Stereographic projection



1. the only conformal mapping of sphere onto a plane which maps circle to circle

Mercator Mapping.

$$ds^2 = [r^2 \cos^2 u^2 (du^1)^2 + (du^2)^2] \quad (\text{1st-FF})$$
$$= r^2 \cos^2 u^2 \left[(du^1)^2 + \left(\frac{1}{\cos u^2}\right)^2 (du^2)^2\right]$$

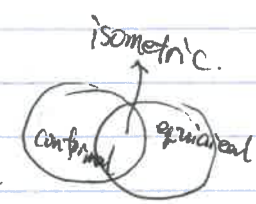
Set $dx_1^* = du^1$ $dx_2^* = \dfrac{du^2}{\cos u^2}$.

$$\Rightarrow x_1^* = u^1 \quad x_2^* = \log \tan \left(\frac{u^2}{2} + \frac{\pi}{4}\right).$$

Their 1st-FF are propotional $\Rightarrow$ conformal.

Equiareal Mapping.

~~Def~~ $\boxed{\text{Theorem}}$. equiareal iff $g = g^*$.
1. Every isometric mapping is equiareal
2. Equiareal and conformal mapping is isometric.



Now consider equiareal mapping from revolution to planes.

$\boxed{\text{Theorem}}$. revolution $C: x_1 = p(u^2)$ $x_2 = 0$ $x_3 = q(u^2)$

$$x(u^1, u^2) = (p(u^2)\cos u^1, p(u^2)\sin u^1, q(u^2))$$

A mapping $x^* = x_1^*(u^1, u^2), x_2^* = x_2^*(u^1, u^2)$ of S $\longrightarrow$ S^* (cartisian) is equiareal iff corrsp Jacobian $\bar{D}$:

$$\bar{D}^2 = p^2(p'^2 + q'^2) \qquad ' : \text{derivative w } u^2.$$

Proof. ~~of~~ consider $g_{\alpha\beta}$: $g_{11} = p^2.\ g_{12} = 0\ \ g_{22} = p'^2 + q'^2$

$\Rightarrow g = p^2(p'^2 + q'^2)$.

S^* 1st-FF of $x^* x^* = 1$,

by introducing $u^1 . u^2$ to S^*, 1 is multiplied by $g^{-}\bar{D}^{2}$.

Equiareal Mapping of sphere into Planes.

<u>Lambert Projection</u>. Sphere $p(u^2) = r\cos u^2$. $q(u^2) = r\sin u^2$.

$\bar{D}$ ~~to~~ must be $\pm r^2 \cos u^2$.

$\hookrightarrow x_1^* = ru^1\ \ x_2^* = r\sin u^2$.

Sanson: $x_1^* = ru^1 \cos u^2\ \ x_2^* = ru^2$.

Bonne: $r^* = r(\frac{1}{2}\pi - u^2)\ \ \alpha^* = \dfrac{u^1\cos u^2}{\frac{1}{2}\pi - u^2}$

Conformal Mapping of the Euclidean Space

(as opposed to mapping surface)

$y_i = h_i(x_1, x_2, x_3)\quad i = 1.2.3$.

Conformal mapp of space ~~is~~ differs in many way with the surface.

Space	Surface
image of a sphere is a sphere, very few conformal map. $\hookrightarrow$ can all be decomposed into inversions: $y = r^2 \dfrac{x^2}{x \cdot x}$	in general, circle $\not\to$ circle. any regular function $f(z)$ $\in$ conformal

in general, orthogonal traj don't exist

$\boxed{\text{Def}}$ Inversion map: $\vec{y} = r^2 \dfrac{\vec{x}^2}{x \circ x}$

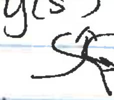
Remark, 1. $f(f(x)) = x$ (thus inversion).

2. fixed points are "unit" sphere.

3. Origin maps to R^∞

4. Inversion is conformal.

Proof: S_x: $x(s)$, $\bar{x}(\sigma)$. $\xrightarrow{f}$ S_y $y(s^*)$ $\bar{y}(\sigma^*)$



to prove 'angle of intersection are ~~same~~ equal: $x_s \cdot \bar{x}_\sigma(\sigma)$

$y_{s^*} = \dfrac{dy}{ds^*} = \dfrac{dy}{ds}\dfrac{ds}{ds^*} = y_s \dfrac{ds}{ds^*}$ $\quad = y_{s^*} \cdot \bar{y}_{\sigma^*}$

$= \dfrac{d}{ds}\dfrac{x}{x \cdot x}\dfrac{ds}{ds^*}$ ......

代入计算

$\boxed{\text{Def}}$ Triply Orthogonal System : 3 surfaces passing through any point are orthogonal. i.e. $x_j \cdot \vec{x}_k = 0$

coordinate surface : $u^i = \text{const}$ $(i = 1, 2, 3)$.

$(u/v = \text{const})$. $((u,w) \text{ or } (v,w))$

$\boxed{\text{Theorem}}$ (Dupin) The <u>curve of</u> intersection of any pair of surface of a triply orthogonal system is a line of curvature on both surfaces

Proof : consider $w = u^3 = \text{const}$, then u, v are the coordinates on the surface w. (w).

will prove : $u/v = \text{const}$ are lines of curvature,

$\Leftrightarrow$ $g_{12} = x_1 \cdot x_2 = 0$, $b_{12} = \vec{x}_{12} \cdot \vec{n} = 0$.

g_{12} : by def of triply otho sys

b_{12} : diff $\frac{d}{dx_k} x_i \cdot x_j = 0$.

$\Rightarrow$ $x_{12} \cdot x_3 = 0$, note $x_3 = x_1 \times x_2 = \vec{n}$.

$\boxed{\text{Theorem}}$ Liouville : f : conformal map of Euclidean space, then sphere $\xrightarrow{f}$ sphere.

$\boxed{\text{Theorem}}$ Every admissible conformal map of space is a composition of (at most 5) inversions.